

# The continuous nonstationary Gabor transform on LCA groups with applications to representations of the affine Weyl-Heisenberg group

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## Abstract

In this paper we introduce and investigate the concept of reproducing pairs which generalizes continuous frames. We will introduce a concept that represents a unifying way to look at certain continuous frames (resp. reproducing pairs) on LCA groups, which can be described as continuous nonstationary Gabor systems and investigate conditions for these systems to form a continuous frame (resp. reproducing pair). As a byproduct we identify the structure of the frame operator (resp. resolution operator). Moreover, we ask the question, whether there always exist mutually dual systems with the same structure such that the resolution operator is given by the identity, i.e. given  $A : X \rightarrow B(\mathcal{H})$ , if there exist  $\psi, \varphi \in \mathcal{H}$ , s.t.

$$f = \int_X \langle f, A(x)\psi \rangle A(x)\varphi d\mu(x), \quad \forall f \in \mathcal{H}$$

and show that the answer is not affirmative. As a counterexample we use a system generated by a unitary action of a subset of the affine Weyl-Heisenberg group in  $L^2(\mathbb{R})$ .

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# 1 Introduction

Motivated by physical applications [2, 23], in order to generalize the coherent states approach, the concept of continuous frames has been introduced in the early 1990's independently by Ali et al. [1] and Kaiser [24]. Coherent states, see e.g. [3], are widely used in many areas of theoretical physics, in particular in quantum mechanics, where classical coherent states are generated by a group action on a single mother wavelet and lead to a resolution of the identity. More general, continuous frames yield a resolution of a positive, bounded and invertible operator.

This raises the question if such a resolution necessitates the frame property. To answer this issue, we will introduce the idea of a reproducing pair where a pair of mappings, in place of a single mapping, is used for an invertible analysis/ synthesis process and compare reproducing pairs to continuous frames. For discrete frames this question of dual systems is a current topic of research, although this is bounded in most cases to the frame property [11], and often to certain types of frames, see e.g. [30, 13]. Introducing reproducing pairs, we ask the more general question of whether reconstruction is possible, without assuming the frame property a-priori. This is related to the topic of frame multipliers [8, 9]. These are operators consisting of analysis, element-wise multiplication with a fixed symbol, and synthesis, and appear in a lot of scientific disciplines. In Physics they represent the link between classical and quantum mechanics, so called quantization operators [3]. The invertibility of multipliers is a central topic [32, 33, 34] in the mathematical investigation of these operators. This includes the question, when a system of two mappings forms a reproducing pair.

Tight frames, i.e. systems where the corresponding frame operator is a multiple of the identity, are often preferred to non-tight ones since the inversion of the frame operator is straightforward. This begs the question if, given a particular structure, there always exist a tight frame, or reproducing pair with resolution operator  $\lambda I$ , with this structure. This will be answered in this paper for the particular class of continuous systems investigated here.

Two of the most widely used continuous frame transforms are the Short Time Fourier Transform (STFT) [19] and the Wavelet transform [28], in particular in signal processing and acoustics. In those applications they are used in their sampled, discretized version. Both transforms have a time-frequency resolution, which is either fixed for all frequencies (for the STFT), or follow a given rule (for wavelets). In practice functions or signals often show particular time-frequency characteristics which call for adaptive and adaptable representation [7]. In [6] the authors introduced adaptivity either in time or frequency, where perfect reconstruction is still possible. This ansatz will be adapted within this paper to introduce continuous nonstationary Gabor frames. We will see in Section 3 that the transforms considered here can be rewritten in terms of a convolution and recall that the Fourier transform

diagonalizes convolutions (within the right function spaces). Hence it seems natural to consider locally compact abelian (LCA) groups as domain of definition, as Fourier theory provides sufficient results on these groups and the Hilbert space  $\mathcal{H} = L^2(G)$ . Within this setting we will then derive a sufficient substitute for the frame (resp. reproducing pair) condition independent of  $f \in L^2(G)$  and show that the frame (resp. resolution) operator is given by a Fourier multiplier.

The study of representations of the affine Weyl-Heisenberg group is of particular interest since it contains both the Weyl-Heisenberg group and the affine group, the underlying groups of the short-time Fourier transform the continuous wavelet transform. Recently a lot of effort has been put in the study of intermediate transforms as the  $\alpha$ -transform, see [14, 15]. The continuous nonstationary Gabor transform is a more general concept, also including this setting, giving rise to a wide range of time frequency transforms. This will give a counterexample to the question whether there always exist a reproducing pair with a given structure such that the resolution operator is the identity.

The present paper is organized as follows: in Section 2 we will briefly present the basic results on Fourier analysis on LCA groups and continuous frames and introduce the concept of reproducing pairs. Section 3 is concerned with the continuous nonstationary Gabor transform on LCA groups. The results are then applied in Section 4 to representations of subsets of the affine Weyl-Heisenberg group and we will show particularities of reproducing pairs in comparison to continuous frames with the help of a particular example.

## 2 Preliminaries

For a short and self-contained introduction to continuous frames, see [31].

**Definition 1** *Let  $\mathcal{H}$  be a Hilbert space and  $(X, \mu)$  be a measure space. A mapping  $\Psi : X \rightarrow \mathcal{H}$  is called a continuous frame if*

- (i)  *$\Psi$  is weakly measurable, i.e.  $x \mapsto \langle f, \Psi(x) \rangle$  is a measurable function for all  $f \in \mathcal{H}$*
- (ii) *there exist positive constants  $A, B > 0$  s.t.*

$$A \|f\|_{\mathcal{H}}^2 \leq \int_X |\langle f, \Psi(x) \rangle|^2 d\mu(x) \leq B \|f\|_{\mathcal{H}}^2, \quad \forall f \in \mathcal{H} \quad (1)$$

The mapping  $\Psi$  is called Bessel if the second, upper inequality in (1) is satisfied.

The standard setting for frames is the discrete version, see for example [12], which is a specialization of this definition and can be reached by choosing  $X$  to be a countable set and  $\mu$  the counting measure. Throughout this paper we will assume that  $\Psi$  is uniformly bounded, i.e.  $\sup_{x \in X} \|\Psi(x)\|_{\mathcal{H}} \leq C$ .

Let us define the basic operators in frame theory, the analysis operator

$$V_{\Psi} : \mathcal{H} \rightarrow L^2(X, \mu), \quad V_{\Psi}f(x) := \langle f, \Psi(x) \rangle$$

and its adjoint operator, called synthesis operator

$$V_{\Psi}^* : L^2(X, \mu) \rightarrow \mathcal{H}, \quad V_{\Psi}^*\varphi := \int_X \varphi(x)\Psi(x)d\mu(x)$$

where the integral is to be understood in the weak sense. By composition of  $V_{\Psi}$  and  $V_{\Psi}^*$  we obtain the frame operator

$$S_{\Psi} : \mathcal{H} \rightarrow \mathcal{H}, \quad S_{\Psi}f := V_{\Psi}^*V_{\Psi}f = \int_X \langle f, \Psi(x) \rangle \Psi(x)d\mu(x)$$

The frame operator is obviously self-adjoint and the frame bounds guarantee that it is positive, bounded and invertible. The mapping  $S_{\Psi}^{-1}\Psi$  is also a frame, called the canonical dual frame, with frame bounds  $B^{-1}, A^{-1}$ .

The analysis operator  $V_{\Psi}$  is in general not onto  $L^2(X, \mu)$  but satisfies a reproducing kernel equation: for  $F \in L^2(X, \mu)$ , there exists  $f \in \mathcal{H}$ , s.t.  $F(x) = V_{\Psi}f(x)$  if and only if  $F(x) = R(F)(x)$  where  $R$  is an integral operator with kernel  $\mathcal{R}(x, y) := \langle S_{\Psi}^{-1}\Psi(y), \Psi(x) \rangle$  and

$$R(F)(x) := \int_X \mathcal{R}(x, y)F(y)d\mu(y)$$

$R$  is moreover the orthogonal projection operator onto the image of  $V_{\Psi}$ .

**Theorem 1** *Let  $\Psi$  be a frame, then the following inversion formula holds*

$$f = \int_X \langle f, \Psi(x) \rangle S_{\Psi}^{-1}\Psi(x)d\mu(x) = \int_X \langle f, S_{\Psi}^{-1}\Psi(x) \rangle \Psi(x)d\mu(x), \quad \forall f \in \mathcal{H} \quad (2)$$

If  $\Psi_d$  is another frame satisfying

$$f = \int_X \langle f, \Psi(x) \rangle \Psi_d(x)d\mu(x) = \int_X \langle f, \Psi_d(x) \rangle \Psi(x)d\mu(x), \quad \forall f \in \mathcal{H}$$

then  $\Psi_d$  is called a dual frame. In most cases there exist several dual frames since in general  $\ker(V_{\Psi}^*) \neq \{0\}$ .

If we do not restrict to an expansion via mutually dual frames we can introduce the following definition motivated by [35, Definition 1.1.1.33]. To this end, we denote the space of bounded linear operators with bounded inverse from  $\mathcal{H}$  to  $\mathcal{K}$  by  $GL(\mathcal{H}, \mathcal{K})$ .

**Definition 2** Let  $(X, \mu)$  be a measure space and  $\Psi, \Phi : X \rightarrow \mathcal{H}$  weakly measurable. The pair of mappings  $(\Psi, \Phi)$  is called a reproducing pair for  $\mathcal{H}$  if the resolution operator  $C_{\Psi, \Phi} : \mathcal{H} \rightarrow \mathcal{H}$ , weakly defined by

$$C_{\Psi, \Phi} f := \int_X \langle f, \Psi(x) \rangle \Phi(x) d\mu(x) \quad (3)$$

is an element of  $GL(\mathcal{H})$ .

Note that this definition is indeed a generalization of continuous frames because if we choose  $\Psi = \Phi$ , then the above definition of reproducing pair corresponds to the original definition of a continuous frame in [1].

In the same paper the authors showed that it is possible to generate new continuous frames “equivalent” to a given continuous frame. In this paper, we will adapt a more general concept from [22] to show the result for reproducing pairs.

**Lemma 2** Let  $\mathcal{K}$  be a Hilbert space,  $(Y, \mu')$  be a measure space,  $\rho : Y \rightarrow X$  a bijective mapping which satisfies  $\mu' \circ \rho^{-1} = \mu$  and preserves measurability,  $T \in GL(\mathcal{H}, \mathcal{K})$  and  $\tau : Y \rightarrow \mathbb{C}$  a measurable function with  $|\tau(y)| = 1$ . If we define  $\tilde{\Psi}(y) := \tau(y)T(\Psi \circ \rho)(y)$  (and  $\tilde{\Phi}$  respectively), then  $(\Psi, \Phi)$  is a reproducing pair for  $\mathcal{H}$  with respect to  $(X, \mu)$ , if and only if  $(\tilde{\Psi}, \tilde{\Phi})$  is a reproducing pair for  $\mathcal{K}$  with respect to  $(Y, \mu')$ .

**Proof:** Let  $f, g \in \mathcal{K}$ . It holds

$$\begin{aligned} \langle C_{\tilde{\Psi}, \tilde{\Phi}} f, g \rangle_{\mathcal{K}} &= \int_Y \langle f, \tau(y)T\Psi(\rho(y)) \rangle_{\mathcal{K}} \langle \tau(y)T\Phi(\rho(y)), g \rangle_{\mathcal{K}} d\mu'(y) \\ &= \int_Y \langle T^* f, \Psi(\rho(y)) \rangle_{\mathcal{H}} \langle \Phi(\rho(y)), T^* g \rangle_{\mathcal{H}} d\mu'(y) \\ &= \int_X \langle T^* f, \Psi(x) \rangle_{\mathcal{H}} \langle \Phi(x), T^* g \rangle_{\mathcal{H}} d\mu(x) \\ &= \langle C_{\Psi, \Phi} T^* f, T^* g \rangle_{\mathcal{H}} \\ &= \langle TC_{\Psi, \Phi} T^* f, g \rangle_{\mathcal{K}} \end{aligned}$$

Hence, we can identify the resolution operator  $C_{\tilde{\Psi}, \tilde{\Phi}} = TC_{\Psi, \Phi} T^*$  and the result follows as  $C_{\tilde{\Psi}, \tilde{\Phi}} \in GL(\mathcal{K})$  if and only if  $C_{\Psi, \Phi} \in GL(\mathcal{H})$ .  $\square$

Unlike the frame operator  $S_{\Psi}$ ,  $C_{\Psi, \Phi}$  is in general neither positive nor self-adjoint, since  $C_{\Psi, \Phi}^* = C_{\Phi, \Psi}$ . From (3) we can also derive a necessary condition for  $F : X \rightarrow \mathbb{C}$  to be an element of the image of  $V_{\Psi}$  in terms of a reproducing kernel. To do so, we need to define the domain of  $V_{\Phi}^*$  as follows

$$\text{dom}(V_{\Phi}^*) := \left\{ F \in L^{\infty}(X, \mu) : \int_X F(x) \Phi(x) d\mu(x) \text{ converges weakly} \right\}$$

**Proposition 3** *Let  $(\Psi, \Phi)$  be a reproducing pair for  $\mathcal{H}$  and  $F \in \text{dom}(V_\Phi^*)$ . It holds that  $F(x) = \langle f, \Psi(x) \rangle$ , for some  $f \in \mathcal{H}$ , if and only if  $F(x) = R(F)(x)$  with the integral kernel*

$$\mathcal{R}(x, y) = \langle C_{\Psi, \Phi}^{-1} \Phi(y), \Psi(x) \rangle$$

*Moreover,  $L^1(X, \mu) \cap L^\infty(X, \mu) \subset \text{dom}(V_\Phi^*)$ , which in particular implies that  $\text{dom}(V_\Phi^*) \cap L^2(X, \mu)$  is dense in  $L^2(X, \mu)$ .*

**Proof:** Let  $F(x) = \langle f, \Psi(x) \rangle$ , then

$$\begin{aligned} R(F)(x) &= \int_X \langle f, \Psi(y) \rangle \langle \Phi(y), (C_{\Psi, \Phi}^{-1})^* \Psi(x) \rangle d\mu(y) \\ &= \langle C_{\Psi, \Phi} f, (C_{\Psi, \Phi}^{-1})^* \Psi(x) \rangle = V_\Psi f(x) = F(x) \end{aligned}$$

Assume now that  $R(F)(x) = F(x)$ . Since,  $F \in \text{dom}(V_\Phi^*)$  we set  $g$  to be the weak limit of  $\int_X F(x) \Phi(x) d\mu(x)$  in  $\mathcal{H}$ . It then follows that  $F(x) = V_\Psi f(x)$ , where  $f := C_{\Psi, \Phi}^{-1} g$ , since

$$\begin{aligned} V_\Psi f(x) &= \langle C_{\Psi, \Phi}^{-1} g, \Psi(x) \rangle = \langle g, (C_{\Psi, \Phi}^{-1})^* \Psi(x) \rangle \\ &= \int_X F(y) \langle \Phi(y), (C_{\Psi, \Phi}^{-1})^* \Psi(x) \rangle d\mu(y) = R(F)(x) = F(x) \end{aligned}$$

It remains to show that if  $F \in L^1(X, \mu) \cap L^\infty(X, \mu)$  it follows that the integral  $\int_X F(x) \Phi(x) d\mu(x)$  converges weakly. Let  $h \in \mathcal{H}$

$$\begin{aligned} \left| \left\langle \int_X F(x) \Phi(x) d\mu(x), h \right\rangle \right| &\leq \int_X |F(x)| |\langle \Phi(x), h \rangle| d\mu(x) \\ &\leq \sup_{x \in X} \|\Phi(x)\|_{\mathcal{H}} \|F\|_{L^1(X, \mu)} \|h\|_{\mathcal{H}} \end{aligned}$$

and hence by Riesz representation theorem  $F \in \text{dom}(V_\Phi^*)$ .  $\square$

Observe that, unlike in the frame setting, there may exist  $f \in \mathcal{H}$ , s.t.  $V_\Psi f \notin L^2(X, \mu)$ , which we will see later on in an example. However, it can be easily seen that the images of  $V_\Psi$  and  $V_\Phi$  are subspaces of mutually dual spaces with the duality pairing  $\langle F, H \rangle = \int_X F(x) \overline{H}(x) d\mu(x)$ .

When dealing with reproducing pairs and continuous frames generated by a particular structure, for example by the action of a group representation to a single window, three questions naturally arise: Are there equivalent or sufficient conditions, independent of  $f \in \mathcal{H}$ , to obtain reproducing pairs or continuous frames? What can be said about the structure of the frame operator (resp. resolution operator)? Given the mapping  $\Psi$  with a particular structure, is there another mapping  $\Phi$  generated by the same structure such that  $(\Psi, \Phi)$  is a reproducing pair and the resolution operator is the identity operator?

In [20] the authors gave a sufficient answer to these questions for the special case that  $X$  is a locally compact group and  $\mu$  the left Haar measure. If  $\pi : G \rightarrow \mathcal{H}$  is a square-integrable group representation, i.e. if it is irreducible and

$$\mathcal{A} := \left\{ \psi \in \mathcal{H} : \int_G |\langle \pi(x)\psi, \psi \rangle|^2 d\mu(x) < \infty \right\} \neq \{0\}$$

then there exists a unique self-adjoint operator  $L$  with domain  $\mathcal{A}$ , s.t. for all  $\psi, \varphi \in \mathcal{A}$  the following orthogonality relation holds

$$\int_G \langle f_1, \pi(x)\psi \rangle \overline{\langle f_2, \pi(x)\varphi \rangle} d\mu(x) = \langle L\varphi, L\psi \rangle \langle f_1, f_2 \rangle$$

Elements of  $\mathcal{A}$  are called admissible windows. If  $G$  is unimodular, then  $L$  is a multiple of the identity. Hence, we see that the resolution operator is the identity after normalization.

Regarding only systems arising from square-integrable group representations is nevertheless rather restrictive. This is why we will introduce more flexible transforms in the next section.

Within this paragraph we list some important results on LCA groups and its Fourier analysis. For a thorough introduction, see the standard text books [17, 27]. The most fundamental examples of LCA groups in harmonic analysis are the additive groups  $\mathbb{R}$ ,  $\mathbb{Z}$ ,  $\mathbb{R}/\mathbb{Z}$  and  $\mathbb{Z}/N\mathbb{Z}$  and their d-fold products. Their relation is depicted in the following diagram.

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\text{Periodization}} & \mathbb{R}/\mathbb{Z} \\ \text{Sampling} \downarrow & & \downarrow \text{Sampling} \\ \mathbb{Z} & \xrightarrow{\text{Periodization}} & \mathbb{Z}/N\mathbb{Z} \end{array}$$

Every LCA group possesses a unique translation invariant measure on  $G$  (up to a constant factor) called the Haar measure, denoted by  $dx$ . Convolution of two functions is given by  $f * g(y) := \int_G f(x)g(x^{-1}y)dx$ . It follows by Riesz-Thorin theorem that convolution with a fixed function  $g \in L^1(G)$  is a bounded operator in  $L^p(G)$ , and  $\|f * g\|_p \leq \|f\|_p \|g\|_1$ , for  $1 \leq p \leq \infty$ .

A character  $\xi$  is a continuous homomorphism from  $G$  to the torus  $\mathbb{T}$ , i.e.  $\xi(xy) = \xi(x)\xi(y)$  and  $|\xi(x)| = 1$ . The dual group  $\widehat{G}$  of  $G$  is the set of all characters of  $G$ . It is an LCA group with pointwise multiplication and the topology of compact convergence on  $G$ . The Pontryagin duality theorem states that any LCA group is “reflexive”, i.e. the dual group of  $\widehat{G}$  is isomorphic to  $G$ . The dual groups of the fundamental examples are given by  $\widehat{\mathbb{R}} \cong \mathbb{R}$ ,  $\widehat{\mathbb{R}/\mathbb{Z}} \cong \mathbb{Z}$ ,  $\widehat{\mathbb{Z}} \cong \mathbb{R}/\mathbb{Z}$  and  $\widehat{\mathbb{Z}/N\mathbb{Z}} \cong \mathbb{Z}/N\mathbb{Z}$ .

Now we are able to define the Fourier transform on  $L^1(G)$  by

$$\widehat{f}(\xi) := \int_G \overline{\xi(x)} f(x) dx, \quad \xi \in \widehat{G}$$

It can be shown that this definition extends to an isometric isomorphism from  $L^2(G)$  to  $L^2(\widehat{G})$  if the Haar measure on  $\widehat{G}$  is appropriately normalized, i.e.  $\|f\|_2 = \|\hat{f}\|_2$  and that Parseval's formula holds, i.e.  $\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle$ ,  $\forall f, g \in L^2(G)$ . In addition, if  $f, g \in L^2(G)$  and  $f * g \in L^2(G)$ , it follows that  $(f * g)^\wedge(\xi) = \hat{f}(\xi)\hat{g}(\xi)$ .

### 3 The continuous nonstationary Gabor transform on LCA groups

With the three questions from the previous section in mind we will now consider continuous systems on  $L^2(G)$  motivated by nonstationary Gabor frames. These discrete systems were introduced by Balazs et al. in [6] in order to gain more flexibility in analyzing signals with specific time-frequency characteristics. A major advantage of nonstationary Gabor frames is that while it allows adaptivity, it still guarantees perfect reconstruction, i.e. resolution of the identity. Even more, under certain conditions, for a family  $\{\psi_n\}_{n \in \mathbb{Z}}$  of compactly supported (resp. band-limited) windows the discrete frame operator is diagonal (resp. diagonal in the Fourier domain) and therefore allows for easy, and consequently fast, inversion. This property has been first studied by Daubechies et al. in [16] and is called the “painless case” in their study.

#### 3.1 Translation invariant systems

Throughout the rest of this paper we assume  $\mathcal{H} = L^2(G)$ , where  $G$  is a second countable LCA group. In particular this assumption implies that  $L^2(G)$  is separable and both  $G$  and  $\widehat{G}$  are  $\sigma$ -compact. As a consequence, it follows that the Haar measures on  $G$  and  $\widehat{G}$  are  $\sigma$ -finite. The translation operator on  $G$  is given by  $T_z f(x) := f(z^{-1}x)$ ,  $x, z \in G$  and its Fourier transform is  $\widehat{T_z f}(\xi) = \xi(z^{-1})\hat{f}(\xi)$ ,  $\xi \in \widehat{G}$ . Now let  $\psi_y, \varphi_y \in L^2(G)$ , for all  $y \in Y$ , where  $(Y, \mu)$  is a measure space with  $\sigma$ -finite measure  $\mu$ . For  $(x, y) \in G \times Y$  we define

$$\Psi(x, y) := T_x \psi_y \quad \text{and} \quad \Phi(x, y) := T_x \varphi_y$$

and the continuous nonstationary Gabor transform (CNSGT) by

$$V_\Psi f(x, y) := \langle f, \Psi(x, y) \rangle$$

Then the following result on the structure of the reproducing (resp. frame) operator holds.

**Theorem 4** *If there exist  $A, B, C > 0$ , s.t.*

$$A \leq |m_{\Psi, \Phi}(\xi)| \leq B, \text{ for a.e. } \xi \in \widehat{G} \tag{4}$$



where

$$m_{\Psi, \Phi}(\xi) := \int_Y \overline{\widehat{\psi}_y(\xi)} \widehat{\varphi}_y(\xi) d\mu(y) \quad (5)$$

and

$$\int_Y |\widehat{\psi}_y(\xi) \widehat{\varphi}_y(\xi)| d\mu(y) \leq C, \text{ for a.e. } \xi \in \widehat{G} \quad (6)$$

then  $(\Psi, \Phi)$  is a reproducing pair for  $L^2(G)$ . The resolution operator is then given weakly by

$$C_{\Psi, \Phi} f = \mathcal{F}^{-1}(m_{\Psi, \Phi} \cdot \mathcal{F}(f)) \quad (7)$$

If  $\Psi = \Phi$ , then  $\Psi$  is Bessel if and only if the upper bound in (4) is satisfied and a continuous frame with frame operator  $S_\Psi = C_{\Psi, \Psi}$  and frame bounds  $A, B$  if and only if condition (4) is satisfied. In particular, the frame is tight if condition (4) becomes an equality.

**Proof:** Let  $f_1, f_2 \in L^1(G) \cap L^2(G)$ ,  $\psi_y, \varphi_y \in L^2(G)$  and assume that (4) and (6) hold. Observe that  $\langle f, T_x \psi_y \rangle = f * \psi_y^*(x)$ , where  $g^*(x) := \overline{g(x^{-1})}$  is the involution of  $g$ . Since  $f \in L^1(G)$  it follows that  $f * \psi_y^* \in L^2(G)$  and therefore  $(f * \psi_y^*)^\wedge(\xi) = \widehat{f}(\xi) \widehat{\psi}_y(\xi)$ . Using this consideration and Parseval's formula we get

$$\begin{aligned} \langle C_{\Psi, \Phi} f_1, f_2 \rangle &= \int_Y \int_G \langle f_1, T_x \psi_y \rangle \overline{\langle f_2, T_x \varphi_y \rangle} dx d\mu(y) \\ &= \int_Y \int_{\widehat{G}} \widehat{f}_1(\xi) \overline{\widehat{f}_2(\xi)} \overline{\widehat{\psi}_y(\xi)} \widehat{\varphi}_y(\xi) d\xi d\mu(y) \\ &= \int_{\widehat{G}} m_{\Psi, \Phi}(\xi) \widehat{f}_1(\xi) \overline{\widehat{f}_2(\xi)} d\xi \\ &= \langle \mathcal{F}^{-1}(m_{\Psi, \Phi} \cdot \mathcal{F}(f_1)), f_2 \rangle \end{aligned}$$

where condition (6) guarantees that Fubini's theorem is applicable. With the usual density argument,  $C_{\Psi, \Phi}$  extends to a continuous operator on  $L^2(G)$  which is bounded with bounded inverse by (4).

It remains to show that if  $\Psi$  is Bessel, it follows that the frame operator is given by (7). By previous calculation we get

$$\langle S_\Psi f, f \rangle = \int_Y \int_{\widehat{G}} |\widehat{f}(\xi)|^2 |\widehat{\psi}_y(\xi)|^2 d\xi d\mu(y) \leq B \|f\|_2^2$$

Consequently, Fubini's theorem is again applicable and the frame operator is given by  $S_\Psi f = \mathcal{F}^{-1}(m_\Psi \cdot \mathcal{F}(f))$ . It is easy to see that  $S_\Psi$  is bounded with bounded inverse only if the symbol  $m_\Psi$  is essentially bounded from above and below.  $\square$

With a slight misuse of terminology we call  $\{\psi_y\}_{y \in Y}$  admissible if (4) is satisfied and  $\{(\psi_y, \varphi_y)\}_{y \in Y}$  cross-admissible, if conditions (4) and (6) are satisfied. Note that the inverse of a Fourier multiplier is given by another Fourier multiplier with the inverse symbol, i.e.  $C_{\Psi, \Phi}^{-1} f = \mathcal{F}^{-1}(m_{\Psi, \Phi}^{-1} \cdot \mathcal{F}(f))$ .

**Corollary 5** *If both  $\Psi$  and  $\Phi$  are Bessel, then  $(\Psi, \Phi)$  is a reproducing system if and only if there exists  $A > 0$ , s.t.  $A \leq |m_{\Psi, \Phi}(\xi)|$ , for a.e.  $\xi \in \widehat{G}$  and  $C_{\Psi, \Phi}$  is given by (7).*

*Suppose that  $C_{\Psi, \Phi}$  is given by (7). The resolution operator is the identity in  $L^2(G)$  if and only if  $m_{\Psi, \Phi}(\xi) = 1$ , for a.e.  $\xi \in \widehat{G}$ .*

*The canonical dual of a translation invariant frame  $\Psi$  is another translation invariant system  $\Phi(x, y) := T_x S_{\Psi}^{-1} \psi_y$*

**Proof:** Let  $\xi \in \widehat{G}$  s.t.  $m_{\Psi}(\xi) \leq B_{\Psi}$  and  $m_{\Phi}(\xi) \leq B_{\Phi}$ , then it follows by Cauchy-Schwarz inequality

$$|m_{\Psi, \Phi}(\xi)| \leq \int_Y |\hat{\psi}_y(\xi) \hat{\varphi}_y(\xi)| d\mu(y) \leq (m_{\Psi}(\xi) m_{\Phi}(\xi))^{1/2} \leq (B_{\Psi} B_{\Phi})^{1/2}$$

Observe that this bound holds for a.e.  $\xi \in \widehat{G}$ , since the set in  $\widehat{G}$  where the bound is violated is a union of two null sets. Consequently, (6) holds and Fubini's theorem is applicable. Finally,  $A \leq |m_{\Psi, \Phi}(\xi)|$ , for a.e.  $\xi \in \widehat{G}$ , guarantees that  $C_{\Psi, \Phi}$  is continuously invertible.

Since the Fourier transform is an onto isometry it follows that the equation  $f = \mathcal{F}^{-1}(m_{\Psi, \Phi}^{-1} \cdot \mathcal{F}(f))$  holds for all  $f \in L^2(G)$  if and only if the Fourier transform of  $f$  is not altered in  $L^2(G)$ , i.e.  $m_{\Psi, \Phi}(\xi) = 1$ , for a.e.  $\xi \in \widehat{G}$ .

To proof the last assertion we only have to show that the translation operator commutes with the inverse frame operator. But this is obviously the case since the inverse frame operator is a Fourier multiplier operator and translation corresponds to character multiplication in Fourier domain.  $\square$

**Remark 6** *This result shows that if  $\Psi, \Phi$  are both Bessel, then  $(\Psi, \Phi)$  is a reproducing pair if the functions  $\psi_y(\xi), \varphi_y(\xi)$  are not orthogonal, or almost orthogonal, in  $L^2(Y, \mu)$  for almost every  $\xi \in \widehat{G}$ .*

### 3.2 Character invariant systems

Now we multiply the windows  $\psi_y$  with a character  $\xi \in \widehat{G}$  instead of translating them, i.e. we consider the operator  $M_{\xi} f(x) := \xi(x) f(x)$  and the mappings

$$\Psi(\xi, y) := M_{\xi} \psi_y \quad \text{and} \quad \Phi(\xi, y) := M_{\xi} \varphi_y$$

where  $(\xi, y) \in \widehat{G} \times Y$  and derive a similar result as in Theorem 4.

**Corollary 7** *The pair of mappings  $(\Psi, \Phi)$  is a reproducing pair for  $L^2(G)$ , if there exist  $A, B, C > 0$  s.t.*

$$A \leq |m_{\Psi, \Phi}(x)| \leq B, \quad \text{for a.e. } x \in G \tag{8}$$

where

$$m_{\Psi, \Phi}(x) := \int_Y \overline{\psi_y(x)} \varphi_y(x) d\mu(y) \tag{9}$$

and

$$\int_Y |\psi_y(x) \varphi_y(x)| d\mu(y) \leq C, \text{ for a.e. } x \in G \quad (10)$$

The resolution operator is weakly given by

$$C_{\Psi, \Phi} f = m_{\Psi, \Phi} \cdot f \quad (11)$$

If  $\Psi = \Phi$ , then  $\Psi$  is Bessel if and only if the upper bound in (8) is satisfied and a continuous frame with frame operator  $S_\Psi = C_{\Psi, \Psi}$  and frame bounds  $A, B$  if and only if condition (8) is satisfied. In particular, the frame is tight if condition (8) becomes an equality.

**Proof:** If one uses that  $\widehat{M_\omega f}(\xi) = T_\omega \hat{f}(\xi)$  which implies  $\langle f, M_\xi \psi_y \rangle = \langle \hat{f}, T_\xi \widehat{\psi_y} \rangle$  and  $\mathcal{F}_{\widehat{G}} \mathcal{F}_G f(x) = f(x^{-1})$  one can follow the proof of Theorem 4 step by step.  $\square$

**Example 1** Let us apply these results to two short examples with  $G = (\mathbb{R}, +)$ . Observe that in this situation  $\widehat{G} \cong G$ . For the short-time Fourier system  $\Psi(x, \omega) = M_\omega T_x \psi$ ,  $\Phi(x, \omega) = M_\omega T_x \varphi$ ,  $(x, \omega) \in \mathbb{R}^{2d}$  with  $\mu$  the Lebesgue measure, one gets the Fourier symbol  $m_{\Psi, \Phi}(\xi) = \langle \varphi, \psi \rangle$  and the well-known inversion formula

$$f = \frac{1}{\langle \varphi, \psi \rangle} \int_{\mathbb{R}^{2d}} \langle f, M_\omega T_x \psi \rangle M_\omega T_x \varphi dx d\omega$$

The second example is chosen to show that the theory also applies to discrete measure spaces with weighted counting measure. Consider the semi-discrete wavelet system with dyadic scale grid, i.e.  $\Psi(x, j) = T_x D_{2^j} \psi$ ,  $(x, j) \in \mathbb{R} \times \mathbb{Z}$ , with the dilation operator  $D_a f(x) := a^{-1/2} f(x/a)$  and  $Y = \mathbb{Z}$  equipped with the weighted counting measure  $\mu(j) = 2^{-j}$ . The Fourier symbol then reads

$$m_\Psi(\xi) = \sum_{j \in \mathbb{Z}} |\hat{\psi}(2^j \xi)|^2$$

It is not difficult to verify that, if  $\hat{\psi}$  is continuous and the following two conditions hold,

$$(i) \exists \xi_0 \neq 0, \text{ s.t. } \inf_{a \in [1, 2]} |\hat{\psi}(a \xi_0)| > 0$$

$$(ii) \exists C > 0, \text{ s.t. } |\hat{\psi}(\xi)|^2 \leq \frac{C|\xi|}{(1+|\xi|)^2}, \forall \xi \in \mathbb{R}$$

$m_\Psi$  is essentially bounded from below by (i) and from above by (ii). The canonical dual frame is another semi-discrete wavelet system. This can be seen if we use Corollary 5 and the observation that  $m_\Psi(2^j \xi) = m_\Psi(\xi)$  for all  $j \in \mathbb{Z}$ .

$$\begin{aligned} S_\Psi^{-1} D_{2^j} \psi &= \mathcal{F}^{-1} (m_\Psi^{-1} D_{2^{-j}} \hat{\psi}) = \mathcal{F}^{-1} (m_\Psi^{-1} (2^j \cdot) D_{2^{-j}} \hat{\psi}) \\ &= \mathcal{F}^{-1} (D_{2^{-j}} (m_\Psi^{-1} \hat{\psi})) = D_{2^j} S_\Psi^{-1} \psi =: D_{2^j} \tilde{\psi} \end{aligned}$$

**Remark 8** *Most of the common continuous transforms used in signal processing can be written as a translation invariant system and can therefore be treated with the previous results. Besides the short-time Fourier transform and the continuous wavelet transform, those are for example the continuous shearlet transform [26] or the continuous curvelet transform [10], just to mention a few.*

*Observe that, although Theorem 4 provides an equivalent condition for finite frame systems on  $\mathbb{C}^N$  it does not supply a condition for discrete frames on  $L^2(\mathbb{R}^d)$ .*

## 4 Reproducing pairs for the affine Weyl-Heisenberg group

Now we want to further reduce the level of abstractness. Recently, greater effort has been put in the study of the affine Weyl-Heisenberg group  $G_{aWH}$  and its representations, see [14, 21, 25, 36, 37], because it contains both the affine group and the Weyl-Heisenberg group as subgroups, the underlying groups of the continuous wavelet transform and the short-time Fourier transform. Consequently, there is a wide range of transforms arising from this group and its subsets.

Topologically  $G_{aWH}$  is isomorphic to  $\mathbb{R}^{2d} \times \mathbb{R}^* \times \mathbb{T}$  with the group law given by

$$(x, \omega, a, \tau) \cdot (x', \omega', a', \tau') = (x + ax', \omega + \omega'/a, aa', \tau \cdot \tau' \cdot e^{-2\pi i \omega' \cdot x/a})$$

with the neutral element  $e = (0, 0, 1, 1)$  and inverse element

$$(-x/a, -a\omega, 1/a, e^{2\pi i \omega \cdot x/a} / \tau)$$

The affine Weyl-Heisenberg group is unimodular and the Haar measure is given by  $d\mu(x, \omega, a, \tau) = dx d\omega |a|^{-1} da d\tau$ . A unitary representation of  $G_{aWH}$  on  $L^2(\mathbb{R}^d)$  is given by

$$\pi(x, \omega, a, \tau)\psi = \tau M_\omega T_x D_a \psi$$

where the basic time-frequency operators on  $\mathbb{R}^d$  are given by

$$T_x f(t) = f(t - x), \quad M_\omega f(t) = e^{2\pi i \omega t} f(t), \quad D_a f(t) = |a|^{-d/2} f(t/a)$$

Since  $G_{aWH}$  is a locally compact group, one is at first interested if this representation is square-integrable. Unfortunately, this is not the case because, loosely speaking, the group is too big.

To overcome this obstacle Torr  sani [36] suggested to regularize the Haar measure by multiplying it with a weight function  $\rho(\omega)$  and showed that under certain conditions this also leads to tight continuous frames. A different approach in the same paper considered subgroups of the affine Weyl-Heisenberg group to obtain square-integrability. For example, if  $d = 1$ , the

section  $(x, \eta_\lambda(a), a, \tau)$  with  $\eta_\lambda(a) = \lambda \left( \frac{1}{a} - 1 \right)$ ,  $\lambda \in \mathbb{R}$  is a subgroup of  $G_{aWH}$  and its representation is square-integrable with left Haar measure  $\frac{dx da}{|a|^2}$  and

$$m_{\Psi, \Phi} \equiv \int_{\mathbb{R}} \overline{\hat{\psi}(\xi)} \hat{\varphi}(\xi) \frac{d\xi}{|\xi + \lambda|}$$

Within the scope of this paper we do not restrict to square-integrable representations but want to use the results from the previous section. The key to reproducing pairs or continuous frames lies in an appropriate restriction of the group parameters and the choice of a measure  $\mu$  on those subsets. Suppose that  $\beta : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $\eta : \mathbb{R}^n \rightarrow \mathbb{R}^d$ , with  $\beta, \eta$  piecewise continuous,  $1 \leq n \leq d$  and  $\{\omega \in \mathbb{R}^n : \beta(\omega) = 0\}$  is a null-set of  $\mathbb{R}^n$ . We consider

$$G_{\beta, \eta} := \left\{ (x, \eta(\omega), \beta(\omega), 1) : (x, \omega) \in \mathbb{R}^{d+n} \right\} \subset G_{aWH}$$

together with the mapping

$$\Psi(x, \omega) = M_{\eta(\omega)} T_x D_{\beta(\omega)} \psi, \quad \psi \in L^2(\mathbb{R}^d)$$

The mapping  $\Psi$  can be rewritten as  $\Psi(x, \omega) = e^{2\pi i x \cdot \eta(\omega)} \tilde{\Psi}(x, \omega)$ , where  $\tilde{\Psi}(x, \omega) = T_x M_{\eta(\omega)} D_{\beta(\omega)} \psi$  is a translation invariant system. By Lemma 2 we can apply the recipe for translation invariant systems from Theorem 4.

As a measure on  $G_{\beta, \eta}$  we take  $d\mu(x, \omega) := d\mu_s(x, \omega) := |\beta(\omega)|^{s-d} dx d\omega$ ,  $s \in \mathbb{R}$ . This particular definition of the measure  $\mu_s$  is justified by two points. For one thing, the behavior of such a system is mainly depending on the scaling function  $\beta$ . For another thing, it is necessary to introduce the parameter  $s$  because altering  $\beta, \eta$  may require a different choices of  $s$  to ensure the existence of continuous frames. To see this we assume that  $d = 1$  and consider on the one hand the continuous wavelet transform, i.e.  $\eta \equiv 0$ ,  $\beta(\omega) = \omega$ . In this case, as  $m_\Psi(\xi) = |\xi|^{-1-s} \int_{\mathbb{R}} |\hat{\psi}(a)|^2 |a|^s da$ , there is no window  $\psi \in L^2(\mathbb{R})$  such that this system forms a continuous frame if  $s \neq -1$ . On the other hand for the setup  $\eta(\omega) = \omega$ ,  $\beta(\omega) = (1 + |\omega|)^{-1}$ , no continuous frame exists if  $s \neq 1$ , which we will explain more in more detail in Example 2. These two short examples also indicate that, in most cases, there is no freedom in the choice of the parameter  $s$  to obtain a frame.

Theorem 4 gives the following two sufficient conditions for  $(\Psi, \Phi)$  to form a reproducing system for  $L^2(\mathbb{R}^d)$

$$A \leq |m_{\Psi, \Phi}(\xi)| \leq B, \text{ for a.e. } \xi \in \mathbb{R}^d$$

where

$$m_{\Psi, \Phi}(\xi) = \int_{\mathbb{R}^n} \overline{\hat{\psi}(\beta(\omega)(\xi - \eta(\omega)))} \hat{\varphi}(\beta(\omega)(\xi - \eta(\omega))) |\beta(\omega)|^s d\omega$$

and

$$\int_{\mathbb{R}^n} \left| \hat{\psi}(\beta(\omega)(\xi - \eta(\omega))) \hat{\varphi}(\beta(\omega)(\xi - \eta(\omega))) \right| |\beta(\omega)|^s d\omega < \infty$$

This result can be found in different articles where special attention is given to particular choices of  $\beta, \eta$ . In [21] composite frames are introduced whereas in [14, 15] a special focus is on the  $\alpha$ -transform and its uncertainty principles. By choosing  $\eta(\omega) = \omega$  we get a transform whose time-frequency resolution is frequency dependent. This is of particular interest for example in audio processing if one wants to construct a transform following the time-frequency resolution of the human auditory system, see [29].

**Example 2** *The following example has been introduced in [21] as composite frames. Let  $d = n = 1$ , and  $\eta(\omega) = \omega$ ,  $\beta(\omega) = (1 + |\omega|)^{-1}$ . Substituting  $z = \beta(\omega)(\xi - \omega)$  yields*

$$\begin{aligned} m_{\Psi, \Phi}(\xi) &= \int_{\mathbb{R}} \overline{\hat{\psi}\left(\frac{\xi - \omega}{1 + |\omega|}\right)} \hat{\varphi}\left(\frac{\xi - \omega}{1 + |\omega|}\right) \frac{d\omega}{(1 + |\omega|)^s} \\ &= |1 + \xi|^{1-s} \int_{-1}^{\xi} \overline{\hat{\psi}(z)} \hat{\varphi}(z) \frac{dz}{(1+z)^{2-s}} + |1 - \xi|^{1-s} \int_{\xi}^1 \overline{\hat{\psi}(z)} \hat{\varphi}(z) \frac{dz}{(1-z)^{2-s}} \end{aligned}$$

*It is easy to see that  $m_{\Psi, \Phi}$  fails to have either upper or lower bound if  $s \neq 1$  and  $m_{\Psi, \Phi}$  reads*

$$m_{\Psi, \Phi}(\xi) = \int_{-1}^{\xi} \overline{\hat{\psi}(z)} \hat{\varphi}(z) \frac{dz}{1+z} + \int_{\xi}^1 \overline{\hat{\psi}(z)} \hat{\varphi}(z) \frac{dz}{1-z} \quad (12)$$

*This expression allows for explicit calculation of  $m_{\Psi, \Phi}$  for many choices of  $\psi, \varphi$ , take for example  $\hat{\psi}(\xi) = (1 - \xi)\chi_A(\xi)$  and  $\hat{\varphi}(\xi) = (1 + \xi)\chi_A(\xi)$ , where  $A := [-1, 1]$ , then one gets  $m_{\Psi, \Phi}(\xi) = 2$ , for  $|\xi| > 1$  and  $m_{\Psi, \Phi}(\xi) = 3 - \xi^2$ , for  $|\xi| \leq 1$ .*

In the following we answer the question whether there always exists a “dual system” with the same structure and examine drawbacks and advantages of reproducing pairs all with the aid of Example 2.

**Proposition 9** *For the transform of Example 2 there is no reproducing pair  $(\Psi, \Phi)$ , s.t.  $\Psi$  and  $\Phi$  are Bessel and  $C_{\Psi, \Phi} = Id_{L^2(\mathbb{R})}$ .*

**Proof:** Since  $\Psi, \Phi$  are Bessel we get by Corollary 5 that  $C_{\Psi, \Phi} = Id_{L^2(\mathbb{R})}$  if and only if  $m_{\Psi, \Phi}(\xi) = 1$ , for a.e.  $\xi \in \mathbb{R}$ . W.l.o.g. we assume that  $\hat{\psi}, \hat{\varphi}$  are real-valued functions. Otherwise use  $Re(m_{\Psi, \Phi})$  instead of  $m_{\Psi, \Phi}$  for the following arguments. In order to obtain a contradiction we assume that there exist  $\psi, \varphi \in L^2(\mathbb{R})$ , s.t.  $m_{\Psi, \Phi} = 1$ , for a.e.  $\xi \in \mathbb{R}$ . Then  $m_{\Psi, \Phi}(\xi) = 1$ , for all  $\xi \in [-1, 1]$ , since both summands in (12) are continuous. Hence, it follows that  $m'_{\Psi, \Phi}(\xi) = 0$ , for every  $\xi \in (-1, 1)$ . On the other hand, Lebesgue’s differentiation theorem states that, for a.e.  $\xi \in (-1, 1)$ , the derivative of  $m_{\Psi, \Phi}$  is given by

$$m'_{\Psi, \Phi}(\xi) = \hat{\psi}(\xi)\hat{\varphi}(\xi) \left[ \frac{1}{1+\xi} - \frac{1}{1-\xi} \right] = -\frac{2\xi}{1-\xi^2} \hat{\psi}(\xi)\hat{\varphi}(\xi)$$

which implies

$$\hat{\psi}(\xi)\hat{\varphi}(\xi) = 0, \text{ for a.e. } \xi \in (-1, 1)$$

This finally yields the contradiction  $m_{\Psi, \Phi}(\xi) = 0, \forall \xi \in [-1, 1]$ .  $\square$

**Proposition 10** *In general it holds that if  $(\Psi, \Phi)$  is a reproducing pair, neither  $\Psi$  nor  $\Phi$  has to be Bessel.*

**Proof:** Consider the pair  $\hat{\psi}(\xi) = (1 - \xi)\chi_A(\xi)$ ,  $\hat{\varphi}(\xi) = (1 + \xi)\chi_A(\xi)$  from Example 2. The Fourier symbol  $m_{\Psi, \Phi}$  is bounded from above and below but neither  $\Psi$  nor  $\Phi$  are Bessel since  $m_{\Psi}$  and  $m_{\Phi}$  are unbounded.  $\square$

Finally, we show that the orbit of  $V_{\Psi}$  possibly contains elements  $V_{\Psi}f \notin L^2(X, \mu)$ . Again, we consider Example 2 with  $\hat{\psi}(\xi) = (1 + \xi)\chi_A(\xi)$  and  $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ ,  $|\hat{f}(\xi)| \geq 1, \forall \xi \in A$ . For every  $(\xi, \omega) \in A \times \mathbb{R}$  it holds  $\beta(\omega)(\xi - \omega) \in A$ . Hence, it follows

$$\begin{aligned} \|V_{\Psi}f\|_{L^2(\mathbb{R}^2, \mu)} &= \int_{\mathbb{R}} \int_{\mathbb{R}} |\hat{f}(\xi)|^2 |\hat{\psi}(\beta(\omega)(\xi - \omega))|^2 \beta(\omega) d\xi d\omega \\ &\geq \int_{\mathbb{R}} \int_A |\hat{\psi}(\beta(\omega)(\xi - \omega))|^2 \beta(\omega) d\xi d\omega \\ &= \int_{\mathbb{R}} \int_A \left(1 + \beta(\omega)(\xi - \omega)\right)^2 \beta(\omega) d\xi d\omega \\ &= \int_{\mathbb{R}} \int_A \left(1 + 2|\omega|\chi_{(-\infty, 0]}(\omega) + \xi\right)^2 \beta(\omega)^3 d\xi d\omega \\ &= \int_{\mathbb{R}} \left[ C_1 + \left(C_2 + 2|\omega|\chi_{(-\infty, 0]}(\omega)\right)^2 \right] \beta(\omega)^3 d\omega \\ &= \infty \end{aligned}$$

## 5 Outlook and discussions

It seems interesting to study discretization schemes for the CNSGT when starting from a semi-discrete system, see Example 1. Clearly, coorbit theory [18] could be applied here but this approach neither exploits that the index set  $Y$  is already discrete, nor the abelian group structure of  $G$ .

Using the results about frame multipliers, the more general concept of reproducing pairs might also be interesting for discrete systems.

Furthermore, as mentioned after Proposition 3, a full characterization of the orbit of  $V_{\Psi}$  is desirable. To this end, it might be worthwhile to construct and investigate Gelfand triples of those spaces, similar to the approach in [4]. Moreover, as the images of  $V_{\Psi}$  and  $V_{\Phi}$  are mutually dual, an investigation in the context of partial inner product spaces [5] seems appropriate.

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